IMPORTANT CALENDAR CHANGES

Close *Tuesday*: 15.3, 15.4

Exam 2 is THURSDAY!!!

10.3,13.4,14.1,14.3,14.4,14.7,15.1-4

(now includes center of mass)

15.1-15.3 Summary

We have 3 ways to describe a region:

"Top/Bottom":

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

"Left/Right":

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

"Inside/Outside":

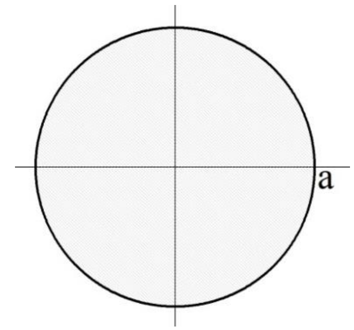
$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta$$

Entry Task:

Let D be the circular disc of radius $r = a$.
Evaluate the integrals.

$$\iint_D 1\,dA = ?$$

$$\iint_D \sqrt{a^2 - x^2 - y^2}\,dA = ?$$



Old Exam Question

Find the area of the region in the first quadrant that is outside $x^2 + y^2 = 2$ and inside the circle $x^2 + y^2 = 2y$.

Old Exam Question

Find the volume of the solid that is inside the cylinder $x^2 + y^2 = 4$, above the plane $z = 1$, and below the surface $z + y = 3 + x^2 + y^2$.

15.4 Center of Mass

Motivation "the see-saw"

New App: Consider a thin plate (*lamina*) with density at each point given by

$\rho(x, y)$ = mass/area (kg/m^2).

We will see that the center of mass (centroid) is given by

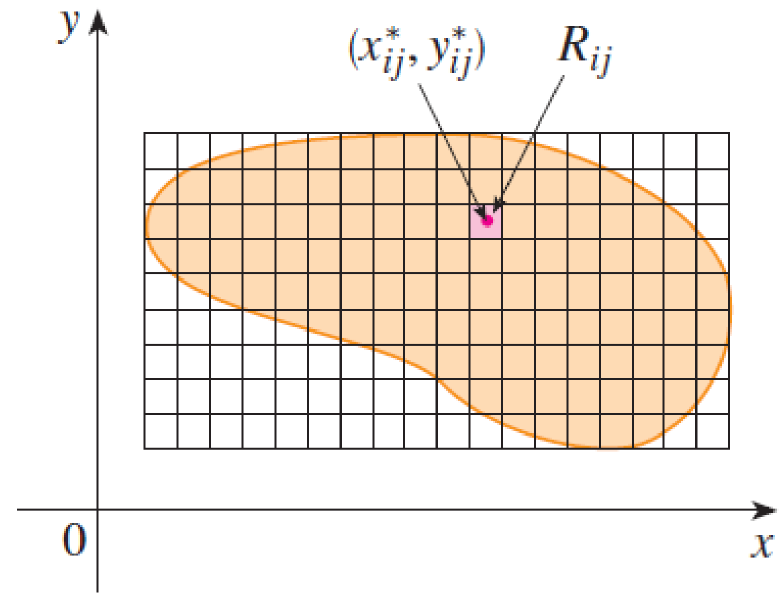
$$\bar{x} = \frac{\text{"Moment about y"}}{\text{Total Mass}} = \frac{\iint_R x\, p(x,y)dA}{\iint_R p(x,y)dA}$$

$$\bar{y} = \frac{\text{"Moment about x"}}{\text{Total Mass}} = \frac{\iint_R y\, p(x,y)dA}{\iint_R p(x,y)dA}$$

In general: If you are given ***n* points** $(x_1,y_1), (x_2,y_2), \ldots, (x_n,y_n)$ with corresponding masses $m_1, m_2, \ldots, m_n$ then

$$\bar{x} = \frac{m_1x_1 + \cdots + m_nx_n}{m_1 + \cdots + m_n} = \frac{M_y}{M}$$

$$\bar{y} = \frac{m_1y_1 + \cdots + m_ny_n}{m_1 + \cdots + m_n} = \frac{M_x}{M}$$



Derivation:

1. Break region into m rows and n columns.
2. Find center of mass of each rectangle:
$$(\bar{x}_{ij}, \bar{y}_{ij})$$
3. Estimate the mass of each rectangle:
$$m_{ij} = p(\bar{x}_{ij}, \bar{y}_{ij})\Delta A$$
4. Now use the formula for *n* points.
5. Take the limit.

$$\bar{x} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}}$$
$$= \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_{ij}\, p(\bar{x}_{ij}, \bar{y}_{ij})\Delta A}{\sum_{i=1}^{m}\sum_{j=1}^{n} p(\bar{x}_{ij}, \bar{y}_{ij})\Delta A}$$

Center of Mass:

$$\bar{x} = \frac{\text{Moment about y}}{\text{Total Mass}} = \frac{\iint_R x\, p(x,y)dA}{\iint_R p(x,y)dA}$$

$$\bar{y} = \frac{\text{Moment about x}}{\text{Total Mass}} = \frac{\iint_R y\, p(x,y)dA}{\iint_R p(x,y)dA}$$

Example:

Consider a 1 by 1 m square metal plate. The density is given by $p(x,y) = kx$ kg/m^2 for some constant k.

Find the center of mass.

Side note:

The density $p(x,y) = kx$ means that the density is proportional to x which can be thought of as distance from the y-axis. In other words, the plate gets heavier at a constant rate from left-to-right.

Translations:

Density proportional to the dist. from…

…the y-axis -- $p(x,y) = kx.$

…the x-axis -- $p(x,y) = ky.$

…the origin -- $p(x,y) = k\sqrt{x^2 + y^2}.$

Density proportional to the square of the distance from the origin:

$$p(x,y) = k(x^2 + y^2).$$

Density inversely proportional to the distance from the origin:

$$p(x,y) = \frac{k}{\sqrt{x^2 + y^2}}$$

Example: A thin plate is in the shape of the region bounded between the circles of radius 1 and 2 in the first quadrant. The density is proportional to the distance from the origin.
Find the center of mass.